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STRUCTURAL COMPUTATIONS WITH THE SINGULAR VALUE DECOMPOSITION OF THE EQUILIBRIUM MATRIX

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Abstract—The Singular Value Decomposition of the equilibrium matrix makes it possible to answer any question of a static, kinematic, or static/kinematic nature for any structural assembly, within a unified computational framework. The paper introduces this decomposition, relates it to the physical properties of a given assembly, and derives formulae for stresses, displacements, etc. A simple example illustrates the calculations.

1. INTRODUCTION

The equilibrium matrix of a structural assembly contains a wealth of static/kinematic information on that assembly. Thus, in addition to the use of the equilibrium matrix in standard structural computations by the Force Method, which is well established (Przemieniecki, 1968; Livesley, 1975; McGuire and Gallagher, 1979; Kaneko *et al.*, 1982), there are many other, more specialized applications. For example, the rank of the equilibrium matrix is needed to investigate the static and kinematic properties of an assembly (Szabo and Kollar, 1984; Pellegrino and Calladine, 1986; Pellegrino, 1988, 1990); the full set of independent states of self-stress, i.e. the nullspace of the equilibrium matrix, is needed in the design of optimal states of prestress, to pretension all cables in a Tensegrity framework (Motro, 1989) or to eliminate the effects of joint backlash in high precision antennas for use in Space (Das *et al.*, 1990); while the kinematic simulation of the motion of a mechanism (Kamman and Huston, 1984; Ider and Amirouche, 1989; Specht, 1987) requires the set of independent zero-energy modes, or independent coordinates, of the system, i.e. the left-nullspace of the equilibrium matrix.

The equilibrium matrix and its transpose, the compatibility matrix, are the cornerstones for all of these (and many other) analyses, but the computations to be performed on them depend on the aim of the analysis, on the type of assembly, e.g. mechanisms cannot take some load conditions without large geometrical distortion, on how sensitive to numerical round-off the results might be, and so on. Thus, in practice the common background is usually lost in radically different algorithmic implementations and, for example, engineers working in the field of deployable structures, i.e. at the interface between structural analysis and mechanisms, often have to use different computer packages to perform parts of essentially the same analysis.

The Singular Value Decomposition (SVD) of the equilibrium matrix makes it possible to answer any question, of a static, kinematic, or static/kinematic nature, within a unified computational framework. Thus, an integrated algorithm for all analyses of all structural assemblies becomes a real possibility. The SVD is well known in Control and, in Mechanics, it has been used by Singh and Likins (1985) for multi-body dynamics. However the rather abstract physical meaning of the reduced variables produced by a full use of SVD was thought to be a drawback of the method by these authors.

The main aim of this paper is to introduce the SVD in the physically-based conceptual framework introduced by Pellegrino and Calladine (1986) and Pellegrino (1990). The layout of the paper is as follows: Section 2 briefly introduces the decomposition for a given structural assembly, while Section 3 relates it to the physical properties of the assembly; Section 4 derives formulae for stresses, displacements, etc.; and Section 5 presents a

simple example. Before doing all this, though, the structural variables and key symbols are introduced.

1.1. *Basic equations*

Given a structural assembly in a particular structural configuration, the *equilibrium matrix* **A**—usually rectangular rather than square—relates the vector $\boldsymbol{\sigma}$ of *generalized stresses*, from which the stress at any point of the structural assembly can be readily calculated, to the vector **l** of *generalized loads*, which can represent any load condition applied to the assembly:

$$\mathbf{A}\boldsymbol{\sigma} = \mathbf{l}. \tag{1}$$

The *generalized strains* $\boldsymbol{\varepsilon}$, assumed to be small, corresponding to the stresses $\boldsymbol{\sigma}$ and the *generalized displacements* **d** from the given configuration, also assumed to be small, corresponding to the loads **l** are related by the *kinematic matrix* **B**:

$$\mathbf{B}\mathbf{d} = \boldsymbol{\varepsilon}. \tag{2}$$

It can be easily shown, by inspection or by Virtual Work, that the equilibrium matrix is equal to the transpose of the kinematic matrix, hence eqn (2) can be written in the form:

$$\mathbf{A}^{\mathrm{T}}\mathbf{d} = \boldsymbol{\varepsilon}. \tag{3}$$

This static–kinematic duality will enable us to perform all of our calculations on data extracted from the equilibrium matrix only or, indeed, from the compatibility matrix only. In general, the coefficients of the equilibrium matrix are functions of the geometrical configuration of the assembly but, since only small displacements from the given configuration are considered in this paper, no updates of **A** are required.

For linear-elastic material behaviour, also assumed throughout this paper, the generalized stresses $\boldsymbol{\sigma}$ are related to the generalized strains $\boldsymbol{\varepsilon}$ by the *square matrix of member flexibilities* **F**, or *flexibility matrix* for short, by the relationship:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \mathbf{F}\boldsymbol{\sigma}. \tag{4}$$

Equations (1), (3) and (4) form the basis of any static, linear-elastic structural analysis.

The choice of suitable sets of variables for each structural element in the assembly, the setting up of the corresponding elemental equilibrium matrices, and the assemblage of these matrices into a global equilibrium matrix are outside the scope of this paper. Standard elements are covered in many textbooks [e.g. Przemieniecki (1968) or McGuire and Gallagher (1979)]; more specialized, recently developed elements for foldable structures can be found in Kwan (1991), Pellegrino *et al.* (1992) and Kwan and Pellegrino (1993). All that is needed for present purposes is the actual equilibrium matrix **A** and, because its dimensions are important, we shall denote its number of rows and columns respectively by n_r and n_c. It follows from eqns (1) and (3) that **l** and **d** have size n_r, while $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have size n_c.

1.2. *Classification*

Since our aim is to set up a computational framework for all types of structural assemblies and for all possible structural analyses, clearly it is important that we are able to identify at the outset the structural properties of each assembly. It is well known (Timoshenko and Young, 1965) that the key parameters which determine the behaviour of any structural assembly are:

m ($\geqslant 0$) number of independent zero-energy deformation modes, or mechanisms. Assemblies with $m = 0$ are *kinematically determinate*;

Table 1. Classification of structural assemblies

Assembly type	Static and kinematic properties		Solution of equilibrium equations (1) and kinematic equations (3)
I	$s = 0$ $m = 0$	Statically determinate and kinematically determinate	Both (1) and (3) have a unique solution for any r.h.s.
II	$s = 0$ $m > 0$	Statically determinate and kinematically indeterminate	(1) has a unique solution for some particular loads, but otherwise no solution. (3) has an m-dimensional infinity of solutions for any strains
III	$s > 0$ $m = 0$	Statically indeterminate and kinematically determinate	(1) has an s-dimensional infinity of solutions for any loads. (3) has a unique solution for compatible strains, but no solution for incompatible strains
IV	$s > 0$ $m > 0$	Statically indeterminate and kinematically indeterminate	(1) and (3) have s-dimensional and m-dimensional infinities of solutions, respectively, for some particular r.h.s., but otherwise admit no solution

s ($\geqslant 0$) number of independent states of self-stress, or degree of hyperstaticity. Assemblies with $s = 0$ are *statically determinate*.

The parameters m and s can be calculated, after determining the rank r of the equilibrium matrix, from the equations (Pellegrino and Calladine, 1986):

$$m = n_r - r \tag{5}$$

and

$$s = n_c - r. \tag{6}$$

The actual values of m and s are not important for our classification scheme: it only matters whether or not they are equal to zero. Table 1 outlines the resulting four types of assemblies; more details can be found in Section 1.6 of Pellegrino (1990).

2. SVD OF THE EQUILIBRIUM MATRIX

For any matrix $\mathbf{A}$ of dimensions $n_r \times n_c$ and rank r, there exist:

- a $n_r \times n_r$ orthogonal matrix $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_{n_r}]$ (by orthogonal matrix we mean that $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, i.e. that the vectors $\mathbf{u}_i$ are orthonormal);
- a $n_c \times n_c$ orthogonal matrix $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_{n_c}]$; and
- a $n_r \times n_c$ matrix $\mathbf{V}$ with r positive elements v_{ii} ($i = 1, \ldots, r$) on the leading diagonal: all other elements are zero;

such that

$$\mathbf{A} = \mathbf{U}\mathbf{V}\mathbf{W}^T. \tag{7}$$

The coefficients v_{ii} are the *singular values* of $\mathbf{A}$ and the vectors $\mathbf{u}_i$ and $\mathbf{w}_i$ are, respectively, the ith *left singular vector* and the ith *right singular vector*. The above Singular Value Decomposition is described in most textbooks on matrix computations, such as Stewart (1973) or Golub and Van Loan (1983). For further details, proofs, etc., the interested reader is referred to these texts.

Considering, for generality, an assembly of Type IV and hence the case $r < \min(n_c, n_r)$, it follows from the above definition that

$$\mathbf{A}\mathbf{w}_i = \begin{cases} \mathbf{v}_{ii}\mathbf{u}_i & i = 1, \ldots, r, \\ \mathbf{0} & i = r+1, \ldots, n_c, \end{cases} \tag{8}$$

$$\mathbf{A}^T\mathbf{u}_i = \begin{cases} \mathbf{v}_{ii}\mathbf{w}_i & i = 1, \ldots, r, \\ \mathbf{0} & i = r+1, \ldots, n_r. \end{cases} \tag{9}$$

Because the right-hand-sides of these equations are, in effect, partitioned, it is convenient to group the left and right singular vectors respectively into the following sub-matrices:

$$\mathbf{U}_r = [\mathbf{u}_1, \ldots, \mathbf{u}_r], \quad \mathbf{U}_{n_r - r} = [\mathbf{u}_{r+1}, \ldots, \mathbf{u}_{n_r}]$$

and

$$\mathbf{W}_r = [\mathbf{w}_1, \ldots, \mathbf{w}_r], \quad \mathbf{W}_{n_c - r} = [\mathbf{w}_{r+1}, \ldots, \mathbf{w}_{n_c}].$$

Thus, $\mathbf{U} = \mathbf{U}_r|\mathbf{U}_{n_r - r}$ and $\mathbf{W} = \mathbf{W}_r|\mathbf{W}_{n_c - r}$. A graphical illustration of the resulting decomposition of **A** is shown in Fig. 1.

There is a strong link between the SVD of **A** and the eigenvalues/eigenvectors of the square, symmetric matrices $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$. It can be shown (Stewart, 1973) that:

(i) $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ have the same set of non-zero eigenvalues and the singular values of **A** are the square roots of these eigenvalues;
(ii) the left singular vectors of **A** are eigenvectors $\mathbf{A}\mathbf{A}^T$, which is the stiffness matrix in the special case of assemblies with $\mathbf{F} = \mathbf{I}$; and
(iii) the right singular vectors of **A** are eigenvectors of $\mathbf{A}^T\mathbf{A}$.

Therefore, the SVD of **A** could be computed by forming these square matrices and computing their eigenvalues and eigenvectors. However, computing the squares of the singular values would result in the loss of significant figures in the smaller singular values, which can play a crucial role in structural computations. Thus, in practice, the standard algorithms for computing the SVD of a matrix **A** perform a series of orthogonal transformations directly on **A** (Golub and Van Loan, 1983). The present paper is not concerned with algorithmic implementations: it will be assumed that a suitable computer routine for computing the SVD of **A** is available. For example, the svd function of Matlab (1989) can be used or, alternatively, the subroutine svdcmp in Press *et al.* (1986).

Once the SVD of the equilibrium matrix has been computed, the value of its rank has to be decided. The problem is that, although the definition of SVD at the beginning of this section states that **V** contains only r non-zero elements, in practice one often finds that the leading diagnonal of **V** contains up to min (n_r, n_c) singular values of decreasing magnitude, none of which is actually equal to zero although some may be quite small. For accurate and stable computations a threshold value α has to be set: all singular values smaller than α are treated as zero, and the rank of **A** is defined accordingly. This approach is also in line with best practice in matrix computations (Golub and Van Loan, 1983).

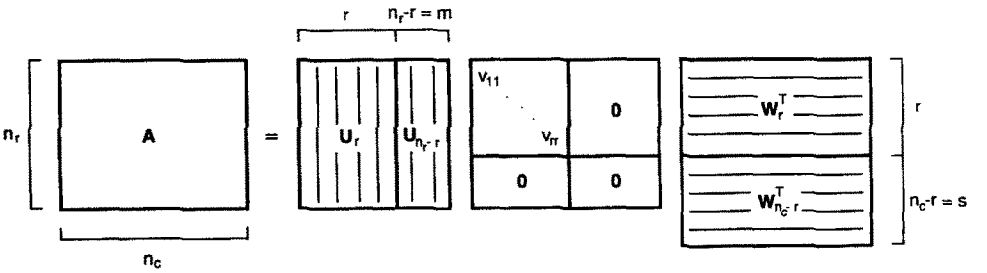

Fig. 1. Graphical illustration of the SVD of the equilibrium matrix $\mathbf{A} = \mathbf{U}\mathbf{V}\mathbf{W}^T$. Note that $m = n_r - r$ and $s = n_c - r$.

For most structural assemblies α can be set to $10^{-3} \cdot v_{11}$ but, near a critical point, i.e. a particular configuration where the rank of the equilibrium matrix drops by one, or more, a smaller α may be required for some calculations. The value of α can be set after the full set of singular values has been calculated and in fact, particularly in difficult cases, its choice can be based upon information provided by the singular values themselves. One of the signs to look for is a wide gap after a fairly continuous range of singular values. Computer accuracy also needs to be considered. In these respects, the SVD is a much more flexible tool that Gaussian elimination type algorithms used by Szabo and Kollar (1984) and by Pellegrino and Calladine (1986), where decisions on the acceptance of small pivots have to be taken during the transformation rather than after it, as in SVD methods.

3. PHYSICAL INTERPRETATION

Equation (8) has a simple static interpretation. The first r equations state that the first r left singular vectors are the load systems in equilibrium with the stress systems in the corresponding right singular vectors, multiplied by the corresponding singular values. Thus we have r orthogonal sets of loads and the corresponding orthogonal stresses. The remaining $s = n_c - r$ equations state that the last s right singular vectors are self-equilibrated stress systems. See Fig. 2(a) for a pictorial representation along the lines of Fig. 1.

Similarly, eqn (9) has a simple kinematic interpretation. The first r equations state that the first r left singular vectors are the displacement systems compatible with the strain systems in the corresponding right singular vectors, divided by the corresponding singular values. Thus, we have r orthogonal sets of displacements and the corresponding orthogonal strains. The remaining $m = n_r - r$ equations state that the last m left singular vectors are strain free, or zero energy, displacement modes [see Fig. 2(b)].

Furthermore, it can be shown that the last m left singular vectors are load conditions which the assembly cannot equilibrate in its current configuration, and also that the last s right singular vectors are orthogonal sets of incompatible strains.

Note that the SVD is an alternative way of finding bases for the four fundamental subspaces of the equilibrium matrix and, although computationally more expensive than the techniques presented in Pellegrino and Calladine (1986), two advantages of the present

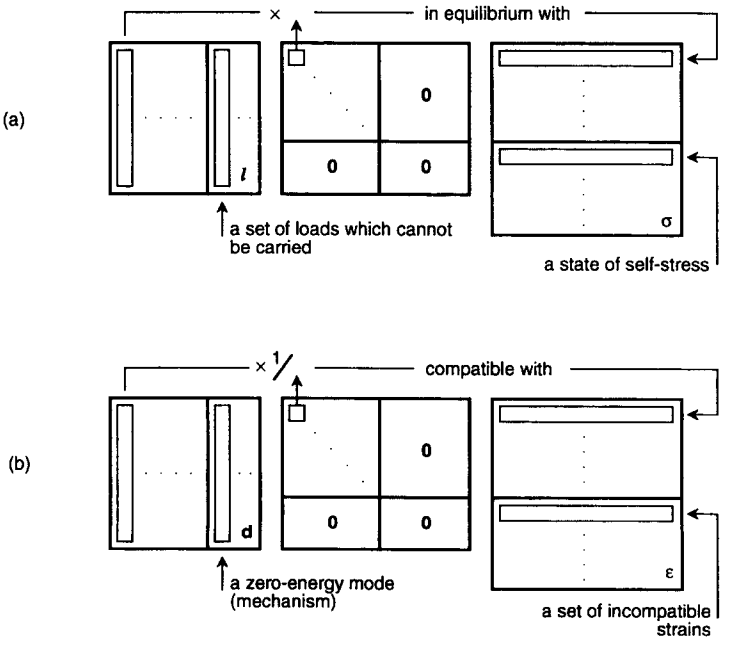

Fig. 2. The first r left and right singular vectors are in a one-to-one correspondence, both in a static (a) and kinematic sense (b). The remaining singular vectors also have static and kinematic interpretations.

approach are that (i) ill-conditioning of the equilibrium matrix poses no problems, as discussed in Section 2, and (ii) the one-to-one static and kinematic correspondence between the left singular vectors in $\mathbf{U}_r$ and the corresponding right singular vectors in $\mathbf{W}_r$, explained above, is very useful for structural computations.

4. FORMULAE FOR STRUCTURAL COMPUTATIONS

Once the SVD of $\mathbf{A}$ has been obtained, any structural computation can be done very quickly and accurately. For generality, we shall derive formulae for a Type IV assembly: formulae for a simpler assembly can be obtained simply by crossing out those terms which do not apply to that particular case.

We compute $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{d}$ due to given loads $\mathbf{l}$ and initial strains $\boldsymbol{\varepsilon}_0$, which is the most common problem in structural analysis. Our formulae, though, can be adapted to solve any other problem. We start by checking that $\mathbf{l}$ can be carried by the assembly, i.e. that the component of $\mathbf{l}$ in the subspace of loads which cannot be carried is zero:

$$\mathbf{U}_{n_r-r}^{\mathrm{T}}\mathbf{l} = \mathbf{0}. \tag{10}$$

If this condition is satisfied, the general solution of eqn (1) is:

$$\boldsymbol{\sigma} = \sum_{i=1}^{r} \frac{\mathbf{u}_i^{\mathrm{T}}\mathbf{l}}{v_{ii}}\mathbf{w}_i + \mathbf{W}_{n_c-r}\mathbf{x}, \tag{11}$$

where the vector $\mathbf{x}$ contains s free parameters. The first term on the right-hand side of eqn (11) is a stress system in equilibrium with $\mathbf{l}$, obtained by finding the components of $\mathbf{l}$ along the first r left singular vectors of $\mathbf{A}$: the stresses due to each of these components are known, Section 3, and are simply superposed. The second term on the right-hand side of eqn (11) is a general state of self-stress: eqn (1) has an s-dimensional infinity of solutions, expressed by the s free components of $\mathbf{x}$.

The value of $\mathbf{x}$ is determined by a set of s compatibility equations: the strains $\boldsymbol{\varepsilon}$ must have a vanishing component in the subspace of incompatible strains, hence

$$\mathbf{W}_{n_c-r}^{\mathrm{T}}\boldsymbol{\varepsilon} = \mathbf{0}. \tag{12}$$

Substituting eqn (11) for $\boldsymbol{\sigma}$ into eqn (4), then into eqn (12), and tidying up, we obtain:

$$\mathbf{W}_{n_c-r}^{\mathrm{T}}\mathbf{F}\mathbf{W}_{n_c-r}\mathbf{x} = -\mathbf{W}_{n_c-r}^{\mathrm{T}}\left(\mathbf{F}\sum_{i=1}^{r}\frac{\mathbf{u}_i^{\mathrm{T}}\mathbf{l}}{v_{ii}}\mathbf{w}_i + \boldsymbol{\varepsilon}_0\right), \tag{13}$$

which is the SVD version of the system of compatibility equations in the standard Force Method of analysis (Pellegrino and Van Heerden, 1990). After solving eqn (13) for $\mathbf{x}$, the final set of stresses is obtained from eqn (11).

Next, the strains are computed from eqn (4), and finally the general solution of eqn (3) is:

$$\mathbf{d} = \sum_{i=1}^{r}\frac{\mathbf{w}_i^{\mathrm{T}}\boldsymbol{\varepsilon}}{v_{ii}}\mathbf{u}_i + \mathbf{U}_{n_r-r}\mathbf{y}, \tag{14}$$

where the m components of $\mathbf{y}$ are free to take any value. The two terms on the right-hand side of eqn (14) are obtained in perfect analogy with eqn (11). A key difference with the static equation (11), though, is that—while a unique solution of eqn (11) can be determined within the linear, small-displacement approach pursued here—this approach cannot discriminate between the m-dimensional infinity of solutions expressed by eqn (14).

Non-linear considerations based on geometry-change effects are required to determine a unique solution (see the next section).

5. AN EXAMPLE

The aim of this section is to illustrate briefly how the method works by means of a simple example. More sophisticated applications will be presented in a forthcoming paper.

Figure 3 shows a two-dimensional pin-jointed assembly, subject to a set of loads. There are no initial strains. The vector of generalized stresses contains the axial forces σ_i in the four bars, and the vector of generalized loads contains the load components in the x- and y-directions at joints 2 and 4. Equation (1) takes the form:

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}.$$

The equilibrium matrix **A** has $n_r = 4$ rows and $n_c = 4$ columns. The vectors of kinematic variables $\boldsymbol{\varepsilon}$ and **d** contain the corresponding bar elongations ε_i and the four displacement components of the joints, respectively. Stresses and strains are related by the flexibility matrix:

$$\mathbf{F} = f \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

where $f = L/CE$, C is the cross-sectional area and E the Young's modulus of the bars.

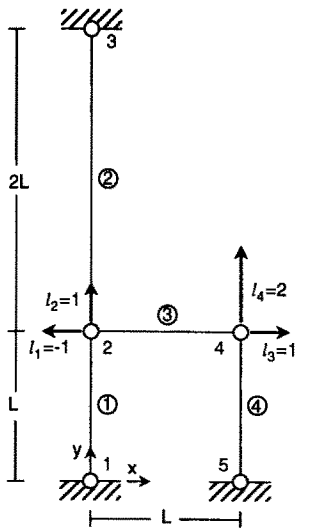

Fig. 3. Two-dimensional pin-jointed assembly analysed in Section 5. All bars have cross-sectional area C and Young's modulus E. For brevity, $f = L/CE$.

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & -0.707 & 0 & -0.707 \\ 1 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & -0.707 \\ 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{ccc|c} 1.414 & 0 & 0 & 0 \\ 0 & 1.414 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cccc} 0.707 & -0.707 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -0.707 & -0.707 & 0 & 0 \end{array}\right]$$

Fig. 4. SVD of the equilibrium matrix for the assembly shown in Fig. 3 (*cf.* Fig. 1).

The SVD of **A**, as obtained from Matlab, is shown in Fig. 4. It can be verified by inspection that this decomposition satisfies the conditions stated at the beginning of Section 2. There are three non-zero singular values, hence $r = 3$, $m = 1$ and $s = 1$: the pin-jointed assembly of Fig. 3 is of Type IV.

This assembly has a single mechanism (and a corresponding set of loads which cannot be equilibrated) which involves the rigid-body translation of bar 3 in the x-direction by an arbitrary, yet small, magnitude (and corresponding equal loads in the x-direction at joints 2 and 4), see the left singular vector $\mathbf{u}_4$. The assembly has a single state of self-stress (set of incompatible elongations) involving equal axial forces in (elongations of) bars 1 and 2, see the right singular vector $\mathbf{w}_4$.

The static and kinematic correspondence between the first three left and right singular vectors can be easily verified. For example, the load condition $\mathbf{u}_1$ which consists of a unit y-load at joint 2, amplified by $\sqrt{2}$ which is the singular value v_{11}, is in equilibrium with axial forces of magnitude $1/\sqrt{2}$, tensile in bar 1 and compressive in bar 2, which form the stress state $\mathbf{w}_1$. From a kinematic viewpoint, the displacement mode $\mathbf{u}_1$ which involves a unit y-displacement of joint 2, divided by the singular value $v_{11} = \sqrt{2}$, is compatible with a $1/\sqrt{2}$ elongation of bar 1 and a $1/\sqrt{2}$ shortening of bar 2, i.e. the strain system $\mathbf{w}_1$.

Next, we consider the particular loads applied to the assembly of Fig. 3 :

$$\mathbf{l} = [-1 \quad 1 \quad 1 \quad 2]^{\mathrm{T}}.$$

We begin by checking that $\mathbf{l}$ is orthogonal to the mechanism, i.e. eqn (10), that

$$\mathbf{u}_4^{\mathrm{T}}\mathbf{l} = [-0.707 \quad 0 \quad -0.707 \quad 0] \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = 0.$$

Obviously, this condition is satisfied and hence, for this particular load, the equilibrium eqn (1) admits a one-dimensional infinity of solutions. Recall that, Table 1, for assemblies of Type IV eqn (1) either admits an s-dimensional infinity of solutions, or none at all. Equation (11) yields:

$$\sigma = \left(\frac{1}{1.414} \begin{bmatrix} 0.707 \\ -0.707 \\ 0 \\ 0 \end{bmatrix} + \frac{1.414}{1.414} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} -0.707 \\ -0.707 \\ 0 \\ 0 \end{bmatrix} x$$

$$= \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -0.707 \\ -0.707 \\ 0 \\ 0 \end{bmatrix} x. \quad (15)$$

The first stress system is in equilibrium with $\mathbf{l}$ while the second term—multiplied by the free parameter x—is self-equilibrated. Note that, because the three non-zero singular values are

of the same order of magnitude, the corresponding contributions to eqn (15) are also of the same magnitude. This would not be the case for geometrically ill-conditioned assemblies.

The value of x is determined by the compatibility equation (13), here:

$$1.5fx = -0.354f,$$

from which $x = -0.236$ and, having substituted this value into eqn (15):

$$\boldsymbol{\sigma} = \begin{bmatrix} 0.667 \\ -0.333 \\ 1 \\ 2 \end{bmatrix}.$$

The strains induced by this stress system are [eqn (4)]:

$$\boldsymbol{\varepsilon} = \mathbf{F}\boldsymbol{\sigma} = f \begin{bmatrix} 0.667 \\ -0.667 \\ 1 \\ 2 \end{bmatrix}.$$

Because $\boldsymbol{\varepsilon}$ is compatible—it can be verified by inspection that $\boldsymbol{\varepsilon}$ is orthogonal to the set of incompatible strains $\mathbf{w}_4$—eqn (3) admits a one-dimensional inifinity of solutions, given by eqn (14):

$$\mathbf{d} = f\left(\frac{0.943}{1.414} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{1.414} \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \\ 0 \end{bmatrix} + \frac{2}{1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} -0.707 \\ 0 \\ -0.707 \\ 0 \end{bmatrix} y$$

$$= f \begin{bmatrix} -0.5 \\ 0.666 \\ 0.5 \\ 2 \end{bmatrix} + \begin{bmatrix} -0.707 \\ 0 \\ -0.707 \\ 0 \end{bmatrix} y. \quad (16)$$

On the basis of the above small-displacement analysis the parameter y would appear to be free to take any value, but actually it is determined by an additional equation based on equilibrium considerations in an infinitesimally displaced configuration, as shown in Section 4 of Pellegrino (1990). Here, the resulting additional equation of orthogonality between the *vector of geometric loads* and $\mathbf{d}$ is

$$[-0.354 \quad 0 \quad -1.414 \quad 0]\mathbf{d} = 0$$

and, substituting eqn (16) for $\mathbf{d}$ and solving for y, we obtain $y = 0.424f$. Thus:

$$\mathbf{d} = f \begin{bmatrix} -0.8 \\ 0.666 \\ 0.2 \\ 2 \end{bmatrix}.$$

6. DISCUSSION

Continuing the work by Pellegrino and Calladine (1986) and Pellegrino (1990), this paper has presented a computational method for the small-displacement analysis of structural assemblies of any type. In principle, any problem could be solved by this approach but, since computing the SVD of a matrix is expensive both in terms of computer time and storage, its use has to be justified by the nature of the problem in hand. For example, it is usually unnecessary for standard stress computations of assemblies which are reliably known to be of Type III.

During the last few years, SVD based algorithms have been used in fast kinematic simulations of the deployment of Space masts (Kwan and Pellegrino, 1990), in the structural analysis of Tensegrity domes (Pellegrino, 1992), and to analyse the folding of closed-section faceted tubes (Guest and Pellegrino, 1993). A related area which is currently being investigated is the accurate computation of special geometrical configurations of kinematic bifurcation, or critical points, where the number of zero-energy modes suddenly increases (Tarnai and Makai, 1989). Here, the problem is that one or more singular values gradually tend to zero, as the assembly approaches the special configuration.

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